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On a continued fraction expansion for Euler's constant

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ABSTRACT

Recently, A.I. Aptekarev and his collaborators found a sequence of rational approximations to Euler's constant γ defined by a third-order homogeneous linear recurrence. In this paper, we give a new interpretation of Aptekarev's approximations in terms of Meijer G-functions and hypergeometric-type series. This approach allows us to describe a very general construction giving linear forms in 1 and γ with rational coefficients. Using this construction we find new rational approximations to γ generated by a second-order inhomogeneous linear recurrence with polynomial coefficients. This leads to a continued fraction (though not a simple continued fraction) for Euler's constant. It seems to be the first non-trivial continued fraction expansion convergent to Euler's constant sub-exponentially, the elements of which can be expressed as a general pattern. It is interesting to note that the same homogeneous recurrence generates a continued fraction for the Euler–Gompertz constant found by Stieltjes in 1895.

1. Introduction

In 1978, R. Apéry [2,18] stunned the mathematical world with a proof that $\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}$ is irrational. Since then many different proofs of this fact have appeared in the literature (see [8] and the references given there). The main idea of all the known proofs is essentially the same and consists in constructing a sequence of linear forms $I_n=u_n\zeta(3)-v_n$, $n=0,1,2,\dots,$ which satisfy the following conditions:

$$\limsup_{n\to\infty}|I_n|^{1/n}\leqslant(\sqrt{2}-1)^4=0.0294372\dots,$$

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$I_n \neq 0$ for infinitely many n, and $u_n \in \mathbb{Z}$, $2D_n^3 v_n \in \mathbb{Z}$, where D_n is the least common multiple of the numbers $1, 2, \dots, n$. If we suppose that $\zeta(3)$ were a rational number a/b, then $2bD_n^3 I_n$ is a non-zero integer for infinitely many n and, on the other hand, it tends to zero as $n \to \infty$ (since $D_n^{1/n} \to e$ and $e^3(\sqrt{2}-1)^4 = 0.591\ldots < 1$), which is a contradiction.

The diversity of all the proposed proofs of the irrationality of $\zeta(3)$ is presented by different interpretations and ways of obtaining the linear forms I_n and its various representations. Apéry showed that the sequences u_n and v_n are given by the formulae

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right)$$

and satisfy the second-order linear recurrence relation

$$(n+1)^3 y_{n+1} - \bigl(34n^3 + 51n^2 + 27n + 5\bigr) y_n + n^3 y_{n-1} = 0$$

with the initial conditions $u_0 = 1$, $u_1 = 5$, $v_0 = 0$, $v_1 = 6$. This implies immediately that v_n/u_n is the nth convergent of the following continued fraction:

$$\zeta(3) = \frac{6}{5} - \frac{1}{117} - \frac{64}{535} - \cdots - \frac{n^6}{34n^3 + 51n^2 + 27n + 5} - \cdots.$$

The shorter proof of Apéry's theorem has been found in 1979 by F. Beukers [5], who used multiple Euler-type integrals and Legendre polynomials.

In 1996 inspired by the works of F. Beukers [6] and L.A. Gutnik [9], Yu.V. Nesterenko [15] proposed another proof of the irrationality of $\zeta(3)$ and a new expansion of this number into continued fraction. His proof was based on the hypergeometric-type series

$$-\frac{1}{2} \sum_{k=1}^{\infty} R_n'(k) = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{d}{dt} \frac{\Gamma^4(t)}{\Gamma^2(t-n)\Gamma^2(t+n+1)} \bigg|_{t=k} = u_n \zeta(3) - v_n \tag{1}$$

that can be written (by the residue theorem) as a complex integral or a Meijer G-function (see [13, Section 5.2], for definition)

$$\begin{aligned} -\sum_{k=1}^{\infty} R_n'(k) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R_n(s) \left(\frac{\pi}{\sin \pi s} \right)^2 ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma^2(n+1-s)\Gamma^4(s)}{\Gamma^2(n+1+s)} \, ds \\ &= G_{4,4}^{4,2} \left(\begin{matrix} -n, -n, n+1, n+1 \\ 0, 0, 0, 0 \end{matrix} \middle| 1 \right), \end{aligned} \tag{2}$$

here c is an arbitrary real number satisfying $0 < c < n+1$ and $R_n(t)$ is a rational function defined by

$$R_n(t) = \frac{(t-1)^2 \cdots (t-n)^2}{t^2 (t+1)^2 \cdots (t+n)^2}.$$

As we can see now, the hypergeometric construction (1), (2) appeared to be more transparent for generalizations on obtaining irrationality results for other odd zeta values (see [19,27]).

Euler's constant was first introduced by Leonhard Euler in 1734 as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)=0.57721566490153286\ldots.$$

It can be considered as an analogue of the value "$\zeta(1)$" of Riemann's zeta function if we compensate the partial sums of the divergent harmonic series by the natural logarithm. It is not known if γ is an irrational or transcendental number. The question of its irrationality remains a famous unresolved problem in the theory of numbers. Even obtaining good rational approximations to it was unknown until recently. First such approximations defined by a third-order linear recurrence were found by A.I. Aptekarev and his collaborators [3] in 2007. More precisely, the numerators $\tilde{p}_n$ and denominators $\tilde{q}_n$ of these approximations are positive integers generated by the recurrence relation

$$\begin{aligned}(16n-15)y_{n+1} &= \left(128n^3+40n^2-82n-45\right)y_n - n^2\left(256n^3-240n^2+64n-7\right)y_{n-1}\\ &\quad + n^2(n-1)^2(16n+1)y_{n-2}\end{aligned} \tag{3}$$

with the initial conditions

$$\begin{aligned}\tilde{p}_0 &= 0, & \tilde{p}_1 &= 2, & \tilde{p}_2 &= 31,\\ \tilde{q}_0 &= 1, & \tilde{q}_1 &= 3, & \tilde{q}_2 &= 50\end{aligned}$$

and the asymptotics

$$\begin{aligned}\tilde{q}_n &= (2n)!\frac{e^{\sqrt{2n}}}{\sqrt[4]{n}}\left(\frac{1}{\sqrt{\pi}(4e)^{3/8}}+O\left(n^{-1/2}\right)\right),\\ \tilde{p}_n-\gamma\tilde{q}_n &= (2n)!\frac{e^{-\sqrt{2n}}}{\sqrt[4]{n}}\left(\frac{2\sqrt{\pi}}{(4e)^{3/8}}+O\left(n^{-1/2}\right)\right).\end{aligned} \tag{4}$$

The remainder of the above approximations is given by the integral [4]

$$\int_0^\infty Q_n(x)e^{-x}\log(x)\,dx = \tilde{p}_n-\gamma\tilde{q}_n, \tag{5}$$

where

$$Q_n(x)=\frac{1}{n!^2}\frac{e^x}{1-x}\left(x^n\left(x^n(1-x)^{2n+1}e^{-x}\right)^{(n)}\right)^{(n)}$$

is a multiple Jacobi–Laguerre orthogonal polynomial on $[0,1]$ and $[1,+\infty)$ with respect to the two weight functions $w_1(x)=(1-x)e^{-x}$, $w_2(x)=(1-x)\log(x)e^{-x}$. The integral (5) can also be written as a multiple integral (see [11, Lemma 4])

$$\int_0^\infty Q_n(x)e^{-x}\log(x)\,dx = \int_0^\infty\int_0^\infty \frac{x^n y^n (x-1)^{2n+1}e^{-x}}{(xy+1)^{n+1}(y+1)^{n+1}}\,dx\,dy. \tag{6}$$

The integrality of the sequences $\tilde{p}_n$ and $\tilde{q}_n$ is not evident and cannot be deduced directly from the recurrence equation (3). Tulyakov [25] proved independently that $\tilde{p}_n$ and $\tilde{q}_n$ are integers, by considering a more "dense" sequence of rational approximations to γ. The present authors (see [10]) found explicit representations for $\tilde{p}_n$ and $\tilde{q}_n$:

$$\tilde{q}_n=\sum_{k=0}^{n}\binom{n}{k}^2(n+k)!,\qquad \tilde{p}_n=\sum_{k=0}^{n}\binom{n}{k}^2(n+k)!(H_{n+k}+2H_{n-k}-2H_k), \tag{7}$$

where $H_n=\sum_{k=1}^{n}1/k$ is the nth harmonic number and $H_0:=0$. Formulae (7) imply that $\tilde{q}_n$ and $\tilde{p}_n$ are integers divisible by $n!$ and $\frac{n!}{D_n}$, respectively. Although the coefficients of the linear forms (4) can be canceled out by the large common factor $\frac{n!}{D_n}$, it is still not enough to prove the irrationality of γ, since the linear γ-forms with integer coefficients:

$$\frac{\tilde{p}_nD_n}{n!}-\frac{\tilde{q}_nD_n}{n!}\gamma\in\mathbb{Z}+\mathbb{Z}\gamma$$

do not tend to zero as n tends to infinity:

$$\frac{\tilde{p}_nD_n}{n!}-\frac{\tilde{q}_nD_n}{n!}\gamma=O\bigl(4^n n^{n-1/4}e^{-\sqrt{2n}}\bigr).$$

Nevertheless, the sequence $\tilde{p}_n/\tilde{q}_n$ provides good rational approximations to Euler's constant

$$\frac{\tilde{p}_n}{\tilde{q}_n}-\gamma=2\pi e^{-2\sqrt{2n}}\bigl(1+O\bigl(n^{-1/2}\bigr)\bigr)\quad \text{as } n\to\infty.$$

In 2009, T. Rivoal [20] found another way of rationally approximating the Euler constant γ, by using multiple Laguerre polynomials

$$A_n(x)=\frac{1}{n!^2}e^x\bigl(x^n\bigl(x^ne^{-x}\bigr)^{(n)}\bigr)^{(n)}.$$

His construction is based on the following third-order recurrence:

$$\begin{aligned}&(n+3)^2(8n+11)(8n+19)y_{n+3}\\&\quad=(n+3)(8n+11)\bigl(24n^2+145n+215\bigr)y_{n+2}-(8n+27)\bigl(24n^3+105n^2+124n+25\bigr)y_{n+1}\\&\qquad+(n+2)^2(8n+19)(8n+27)y_n,\end{aligned}$$

which provides two sequences of rational numbers P_n and Q_n, $n\geqslant 0$, with the initial values

$$\begin{aligned}P_0&=-1, & P_1&=4, & P_2&=77/4,\\ Q_0&=1, & Q_1&=7, & Q_2&=65/2\end{aligned}$$

such that $\frac{P_n}{Q_n}$ converges to γ. The sequences P_n, Q_n satisfy the inclusions

$$n!Q_n,\quad n!D_nP_n\in\mathbb{Z},$$

which were proved in [11, Corollary 5], and provide better approximations to γ,

$$\left|\frac{P_n}{Q_n}-\gamma\right| \leqslant c_0 e^{-9/2n^{2/3}+3/2n^{1/3}}, \qquad |Q_n| = O\left(e^{3n^{2/3}-n^{1/3}}\right) \quad \text{as } n \to \infty.$$

Unfortunately, this convergence is not fast enough to imply the irrationality of γ.

In this paper, we give a new interpretation of Aptekarev's approximations to γ in terms of Meijer G-functions and hypergeometric-type series, which can be considered as an analog of the complex integral (2) and series (1) for Euler's constant. This approach allows us to describe a very general construction giving linear forms in 1 and γ with rational coefficients. Using this construction we find new rational approximations to γ generated by a second-order inhomogeneous linear recurrence with polynomial coefficients. This leads to a continued fraction (though not a simple continued fraction) for Euler's constant. It seems to be the first non-trivial continued fraction expansion convergent to Euler's constant sub-exponentially, the elements of which can be expressed as a general pattern. It is interesting to note that the same homogeneous second-order linear recurrence generates a continued fraction for the Euler–Gompertz constant found by Stieltjes in 1895.

2. Analogs of hypergeometric-type series and complex integrals for Euler's constant

Note that the first attempt to generalize series (1) to find suitable approximations for Euler's constant γ was made by Sondow [23]. He introduced the following series:

$$I_n := \sum_{\nu=n+1}^{\infty} \int_{\nu}^{\infty} \left(\frac{n!}{x(x+1)\cdots(x+n)}\right)^2 dx$$

and proved that

$$I_n = \binom{2n}{n}\gamma + L_n - \sum_{i=0}^{n} \binom{n}{i}^2 H_{n+i} = O\left(2^{-4n} n^{-1}\right), \tag{8}$$

where

$$L_n = 2\sum_{k=1}^{n}\sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i}-H_i)\log(n+k).$$

Unfortunately, this type of approximations contains linear forms in logarithms of rational numbers and this enables only to obtain conditional irrationality criteria for γ that require knowledge of the growth of the fractional parts $\{D_{2n}L_n\}$.

In this section, we obtain new representations for Aptekarev's linear form $\tilde{p}_n - \gamma\tilde{q}_n$ distinct from (5) and (6) which can be considered as analogs of the hypergeometric-type series (1) and complex integral (2). It can be easily done by using explicit formulae (7).

Proposition 1. *For each $n = 0, 1, 2, \ldots,$ the following equality holds*

$$\tilde{f}_n := \tilde{p}_n - \gamma\tilde{q}_n = n!^2 \sum_{k=0}^{n} \frac{d}{dt}\left(\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\right)\bigg|_{t=k}.$$

Proof. The straightforward verification shows that

$$\frac{d}{dt}\left(\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\right)$$
$$=\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\big(\psi(n+t+1)-2\psi(t+1)+2\psi(n-t+1)\big), \tag{9}$$

where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the logarithmic derivative of the gamma function, also known as the digamma function. Summing (9) for $t=0,1,\dots,n$ and using the well-known properties of the digamma function

$$\psi(1)=-\gamma,\qquad \psi(n+1)=H_n-\gamma,\quad n\geqslant 1,$$

we have

$$n!^2\sum_{k=0}^{n}\frac{d}{dt}\left(\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\right)\bigg|_{t=k}$$
$$=n!^2\sum_{k=0}^{n}\frac{\Gamma(n+k+1)}{\Gamma^2(k+1)\Gamma^2(n-k+1)}\big(\psi(n+k+1)-2\psi(k+1)+2\psi(n-k+1)\big)$$
$$=\sum_{k=0}^{n}\binom{n}{k}^2(n+k)!(H_{n+k}-2H_k+2H_{n-k}-\gamma)=\tilde{p}_n-\gamma\tilde{q}_n,$$

as required. □

Proposition 2. *For each $n=0,1,2,\dots,$ we have*

$$\tilde{f}_n+\frac{n!^4}{(2n+1)!^2}\,{}_2F_2\left(\begin{matrix}n+1,n+1\\2n+2,2n+2\end{matrix}\,\middle|\,-1\right)$$
$$=\frac{n!^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\left(\frac{\pi}{\sin\pi t}\right)^2dt$$
$$=\frac{n!^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(n+t+1)\Gamma^2(t-n)}{\Gamma^2(t+1)}\,dt=n!^2G^{0,3}_{3,2}\left(\begin{matrix}-n,n+1,n+1\\0,0\end{matrix}\,\middle|\,1\right), \tag{10}$$

$$\tilde{q}_n=\frac{n!^2}{2\pi i}\int_L\frac{\Gamma(n+t+1)e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\cdot\frac{\pi}{\sin\pi t}\,dt$$
$$=\frac{(-1)^n n!^2}{2\pi i}\int_L\frac{\Gamma(n+t+1)\Gamma(t-n)}{\Gamma^2(t+1)\Gamma(n-t+1)}e^{i\pi t}\,dt$$
$$=(-1)^n n!^2G^{0,2}_{3,2}\left(\begin{matrix}-n,n+1,n+1\\0,0\end{matrix}\,\middle|\,-1\right), \tag{11}$$

where ${}_rF_s$ is the generalized hypergeometric function, c is an arbitrary real number satisfying $c>n$, and L is a loop beginning and ending at $-\infty$ and encircling the points $n,n-1,n-2,\dots$ exactly once in the positive direction.

Proof. We first note that the equality of two complex integrals in (10) and (11) follows easily by the reflection formula for the gamma function:

$$\Gamma(t-n)\Gamma(n-t+1)=\frac{(-1)^n\pi}{\sin\pi t}.$$

The third equality in both formulae (10) and (11) follows by the definition of the Meijer G-function. To prove the first equality in (10), we consider the integrands of the complex integrals (10) on the rectangular contour with vertices $c\pm iN$, $-N-1/2\pm iN$, where N is a sufficiently large integer, $N>c$. Then, by the residue theorem, we have that the integral

$$\frac{1}{2\pi i}\left(\int\limits_{c-iN}^{c+iN}+\int\limits_{c+iN}^{-N-\frac{1}{2}+iN}+\int\limits_{-N-\frac{1}{2}+iN}^{-N-\frac{1}{2}-iN}+\int\limits_{-N-\frac{1}{2}-iN}^{c-iN}\right)\frac{\Gamma(n+t+1)\Gamma^2(t-n)}{\Gamma^2(t+1)}\,dt \tag{12}$$

is equal to the sum of the residues of the integrand at integer points $t=k$, $-N\leqslant k\leqslant n$. Using the expansion

$$\left(\frac{\pi}{\sin\pi t}\right)^2=\frac{1}{(t-k)^2}+O(1)$$

in a neighborhood of the integer point $t=k$ we obtain that the integral (12) is equal to

$$\begin{aligned}&\sum_{k=0}^{n}\operatorname*{res}_{t=k}\left(\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\left(\frac{\pi}{\sin\pi t}\right)^2\right)+\sum_{k=-N}^{-n-1}\operatorname*{res}_{t=k}\left(\frac{\Gamma(n+t+1)\Gamma^2(t-n)}{\Gamma^2(t+1)}\right)\\&\quad=\sum_{k=0}^{n}\frac{d}{dt}\left(\frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\right)\bigg|_{t=k}+\sum_{k=0}^{N-n-1}\frac{(n+k)!^2(-1)^k}{(2n+k+1)!^2k!}.\end{aligned} \tag{13}$$

Since on the sides $[c+iN,-N-1/2+iN]$, $[-N-1/2+iN,-N-1/2-iN]$, $[-N-1/2-iN,c-iN]$ of the rectangle we have $|t|=O(N)$, it follows that

$$\left|\frac{\Gamma^2(t-n)}{\Gamma^2(t+1)}\right|=\frac{1}{|t^2(t-1)^2\cdots(t-n)^2|}=O\left(\frac{1}{N^{2n+2}}\right). \tag{14}$$

For large $|z|$ the asymptotic expansion of the gamma function is [13, Section 2.11]

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+O\left(|z|^{-1}\right), \tag{15}$$

where $|\arg z|\leqslant\pi-\varepsilon$, $\varepsilon>0$ and the constant in O is independent of z. Then for $t=x\pm iN$, $-N\leqslant x\leqslant c$, we have

$$\begin{aligned}\left|\Gamma(n+t+1)\right|&=\left|\Gamma(x+n+1\pm iN)\right|=O\left(e^{(x+n+\frac{1}{2})\log N\mp N\arg(x+n+1\pm iN)}\right)\\&\leqslant O\left(N^{c+n+\frac{1}{2}}e^{-\frac{\pi}{4}N}\right).\end{aligned} \tag{16}$$

On the segment $[-N-1/2+iN,-N-1/2-iN]$ we use the trivial estimate

$$\begin{aligned}\bigl|\Gamma(n+t+1)\bigr| &\leqslant \bigl|\Gamma\bigl(\mathrm{Re}(n+t+1)\bigr)\bigr| = |\Gamma(n+1/2-N)| = \frac{\pi}{\Gamma(N+1/2-n)}\\ &= O\bigl(e^{-N\log N+N}\bigr).\end{aligned}\tag{17}$$

Summarizing (14), (16), (17) and letting N tend to infinity in (12), by (13) and Proposition 1, we get the first equality in (10).

Now we prove the first equality in (11). For this purpose, suppose that C_1 and C_2 are points of intersections of the loop L with the vertical line $\mathrm{Re}(t)=-N-1/2$ and consider a closed contour L^* oriented in the positive direction and consisting of the segment C_1C_2 and the right part of the loop L connecting the points C_1 and C_2, which we denote by $\widetilde{C_1C_2}$. Then, by the residue theorem, we obtain

$$\begin{aligned}&\frac{n!^2}{2i}\int\limits_{L^*}\frac{\Gamma(n+t+1)e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\frac{dt}{\sin\pi t}\\ &\quad = n!^2\sum_{k=0}^{n}\operatorname*{res}_{t=k}\left(\frac{\Gamma(n+t+1)e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\frac{\pi}{\sin\pi t}\right)=\tilde{q}_n.\end{aligned}\tag{18}$$

On the other hand, we have

$$\int\limits_{L^*}=\int\limits_{\widetilde{C_1C_2}}+\int\limits_{C_2C_1}.\tag{19}$$

Since

$$\frac{\Gamma(n+t+1)e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1)}\frac{\pi}{\sin\pi t}=\frac{1}{2\pi i}\frac{\Gamma^2(t-n)}{\Gamma^2(t+1)}\Gamma(n+t+1)\bigl(e^{2i\pi t}-1\bigr)$$

and the function $e^{2i\pi t}-1$ is bounded on the vertical segment C_1C_2, by (14), (17) and (19) we get

$$\int\limits_{L^*}=\int\limits_{\widetilde{C_1C_2}}+O\bigl(e^{-N\log N+N}\bigr).$$

Finally, letting N tend to infinity and taking into account (18), we get the required formula. $\square$

Now, more generally, consider an arbitrary function $F(n,t)$ of the form

$$F(n,t)=\frac{\prod_{j=1}^{s}\Gamma(a_jn+b_jt+1)}{\prod_{j=1}^{u}\Gamma(c_jn+d_jt+1)},\tag{20}$$

where $a_j,b_j,c_j,d_j\in\mathbb{Z}$, $\sum_{j=1}^{s}b_j\neq\sum_{j=1}^{u}d_j$, and all gamma values are well defined for $0\leqslant t\leqslant M(n)$. We say that $\Gamma(an+bt+1)$ is well defined if $an+bt+1$ is not a negative integer or zero.

Proposition 3. *Let $F(n,t)$ be defined as above, $M(n)$ be a non-negative integer, and*

$$F_n=\sum_{t=0}^{M(n)}\frac{d}{dt}F(n,t).$$

Then for each $n = 0, 1, 2, \ldots,$ we have $F_n = p_n - \gamma q_n$ with

$$q_n = \left(\sum_{j=1}^{s} b_j - \sum_{j=1}^{u} d_j \right) \cdot \sum_{k=0}^{M(n)} F(n,k)$$

and

$$p_n = \sum_{k=0}^{M(n)} F(n,k) \left(\sum_{j=1}^{s} b_j H_{a_j n + b_j k} - \sum_{j=1}^{u} d_j H_{c_j n + d_j k} \right).$$

Proof. Differentiating $F(n,t)$ with respect to t and summing over $t = 0, 1, \ldots, M(n)$, we have

$$\begin{aligned}
\sum_{t=0}^{M(n)} \frac{d}{dt} F(n,t) &= \sum_{k=0}^{M(n)} \frac{\prod_{j=1}^{s} \Gamma(a_j n + b_j k + 1)}{\prod_{j=1}^{u} \Gamma(c_j n + d_j k + 1)} \left(\sum_{j=1}^{s} b_j \psi(a_j n + b_j k + 1) - \sum_{j=1}^{u} d_j \psi(c_j n + d_j k + 1) \right) \\
&= \sum_{k=0}^{M(n)} \frac{\prod_{j=1}^{s} (a_j n + b_j k)!}{\prod_{j=1}^{u} (c_j n + d_j k)!} \left(\sum_{j=1}^{s} b_j (H_{a_j n + b_j k} - \gamma) - \sum_{j=1}^{u} d_j (H_{c_j n + d_j k} - \gamma) \right) \\
&= p_n - \gamma q_n. \qquad \square
\end{aligned}$$

3. A second-order inhomogeneous linear recurrence for Euler's constant

In this section, we consider application of Proposition 3 to the function

$$F(n,t) = \frac{\Gamma^2(n+1)}{\Gamma(t+1)\Gamma^2(n-t+1)}, \quad n \in \mathbb{Z},\ n \geqslant 0.$$

Then for

$$F_n := \sum_{t=0}^{n} \frac{d}{dt} F(n,t),$$

we have $F_n = p_n - \gamma q_n$ with

$$q_n = \sum_{k=0}^{n} \binom{n}{k}^2 k!, \qquad p_n = \sum_{k=0}^{n} \binom{n}{k}^2 k! (2H_{n-k} - H_k), \quad n = 0, 1, 2, \ldots. \tag{21}$$

Lemma 1. *The sequence $\{q_n\}_{n=0}^{\infty}$ is a solution of the second-order homogeneous linear recurrence*

$$q_{n+2} - 2(n+2) q_{n+1} + (n+1)^2 q_n = 0 \tag{22}$$

with the initial values $q_0 = 1$, $q_1 = 2$, and the sequence $\{p_n\}_{n=0}^{\infty}$ is a solution of the second-order inhomogeneous linear recurrence

$$p_{n+2} - 2(n+2) p_{n+1} + (n+1)^2 p_n = -\frac{n}{n+2} \tag{23}$$

with the initial values $p_0 = 0$, $p_1 = 1$.

Proof. Applying Zeilberger's algorithm of creative telescoping [17, Chapter 6] to the function $F(n,t)$ we get for each $n=0,1,2,\dots$ the identity

$$F(n+2,t)-2(n+2)F(n+1,t)+(n+1)^2F(n,t)=G(n,t+1)-G(n,t), \tag{24}$$

where

$$G(n,t)=\frac{\Gamma^2(n+2)\cdot r(n,t)}{\Gamma(t+1)\Gamma^2(n-t+3)} \quad \text{and} \quad r(n,t)=t\bigl(t^2-(2n+3)t+n(n+2)\bigr).$$

To prove (24) it is sufficient to multiply both sides of (24) by $\Gamma^2(n-t+3)\Gamma(t+1)/\Gamma^2(n+2)$ and after cancellation of gamma factors to verify the identity

$$(n+2)^2-2(n+2)(n-t+2)^2+(n-t+2)^2(n-t+1)^2=\frac{(n-t+2)^2}{t+1}r(n,t+1)-r(n,t).$$

Summing equality (24) over $t=0,1,2,\dots$ and taking into account that

$$\lim_{t\to k}G(n,t)=0, \quad k=n+3,n+4,\dots,$$

we get the difference equation for q_n:

$$q_{n+2}-2(n+2)q_{n+1}+(n+1)^2q_n=-G(n,0)=0.$$

In order to get the recurrence relation for the sequence p_n, it is convenient to rewrite F_n as an infinite sum

$$F_n:=\sum_{t=0}^{\infty}\frac{d}{dt}F(n,t),$$

taking into account that

$$\lim_{t\to k}\frac{d}{dt}F(n,t)=0 \quad \text{for } k=n+1,n+2,\dots.$$

Then differentiating (24) with respect to t and summing over $t=0,1,2,\dots$ we get for each $n=0,1,2,\dots,$

$$F_{n+2}-2(n+2)F_{n+1}+(n+1)^2F_n=\lim_{k\to\infty}G'(n,k+1)-G'(n,0). \tag{25}$$

Since

$$G'(n,t)=(n+1)!^2\biggl(\frac{t^3-(2n+3)t^2+tn(n+2)}{\Gamma(t+1)\Gamma^2(n-t+3)}\bigl(2\psi(n-t+3)-\psi(t+1)\bigr)$$
$$+\frac{3t^2-2t(2n+3)+n(n+2)}{\Gamma(t+1)\Gamma^2(n-t+3)}\biggr),$$

we see that

$$\lim_{k\to\infty} G'(n,k+1)=0 \quad \text{and} \quad G'(n,0)=\frac{n}{n+2},$$

and consequently, (25) becomes

$$F_{n+2}-2(n+2)F_{n+1}+(n+1)^2F_n=-\frac{n}{n+2}, \quad n=0,1,2,\ldots.$$

This implies that the sequence $p_n=F_n+\gamma q_n$ satisfies the same inhomogeneous recurrence, and the lemma is proved. $\square$

4. Rate of convergence of rational approximations

In this section we show that the sequence p_n/q_n converges to Euler's constant γ and investigate its rate of convergence. We begin with defining a complex integral I_n by means of the Meijer G-function:

$$I_n := n!^2 G^{0,2}_{2,1}\left(\begin{matrix} n+1, n+1 \\ 0 \end{matrix}\,\middle|\, 1\right)=\frac{n!^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma^2(t-n)}{\Gamma(t+1)}\,dt, \tag{26}$$

where $c>n$ is an arbitrary constant.

Lemma 2. *The following formula holds*

$$F_n=I_n+O\left(\frac{1}{n^2}\right) \quad \text{as } n\to\infty,$$

where the constant in O is absolute.

Proof. Since

$$\frac{\Gamma^2(t-n)}{\Gamma(t+1)}=\Gamma(t-n)\left(\frac{\Gamma(t-n)}{\Gamma(t+1)}\right), \tag{27}$$

by a similar argument as in the proof of Proposition 2, considering the integrand (27) on the rectangular contour with vertices $c\pm iN$, $-N-1/2\pm iN$, where N is a sufficiently large integer, we conclude that the integral (26) can be evaluated as a sum of residues at the points $n, n-1, \ldots$. It is easily seen that the function (27) has double poles at the points $0,1,2,\ldots,n$ and simple poles at $-1,-2,\ldots$. Therefore, we have

$$\begin{aligned} I_n &= n!^2\sum_{k=-\infty}^{n}\operatorname*{res}_{t=k}\left(\frac{\Gamma^2(t-n)}{\Gamma(t+1)}\right) \\ &= n!^2\sum_{k=-\infty}^{-1}\operatorname*{res}_{t=k}\left(\frac{\pi}{\sin\pi t}\cdot\frac{1}{\Gamma(n-t+1)\cdot t(1-t)\cdots(n-t)}\right) \\ &\quad + n!^2\sum_{k=0}^{n}\operatorname*{res}_{t=k}\left(\frac{1}{\Gamma(t+1)\Gamma^2(n-t+1)}\left(\frac{\pi}{\sin\pi t}\right)^2\right) \end{aligned}$$

$$= n!^2 \sum_{k=-\infty}^{-1} \frac{(-1)^k}{(n-k)! \cdot k(1-k)\cdots(n-k)}$$

$$+ n!^2 \sum_{k=0}^{n} \frac{d}{dt}\left(\frac{1}{\Gamma(t+1)\Gamma^2(n-t+1)}\right)\Bigg|_{t=k} = \frac{1}{(n+1)^2} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(n+2)_k^2} + F_n.$$

Since

$$\left|\sum_{k=0}^{\infty} \frac{(-1)^k k!}{(n+2)_k^2}\right| \leqslant \sum_{k=0}^{\infty} \frac{k!}{(n+2)_k^2} \leqslant \sum_{k=0}^{\infty} \frac{1}{k!} = e,$$

we get the desired assertion. □

Lemma 3. *The following asymptotic formulae hold*

$$F_n = n! \frac{e^{-2\sqrt{n}}}{n^{1/4}}\left(\sqrt{\frac{\pi}{e}} + O\left(n^{-1/2}\right)\right) \quad \textit{as } n \to \infty,$$

$$q_n = n! \frac{e^{2\sqrt{n}}}{n^{1/4}}\left(\frac{1}{2\sqrt{\pi e}} + O\left(n^{-1/2}\right)\right) \quad \textit{as } n \to \infty.$$

Proof. It is easy to show that the complex integral I_n can be expressed in terms of the Whittaker function $W_{\kappa,\mu}(z)$ which is one of the solutions of Whittaker's confluent hypergeometric equation (see [7, Section 2], [22, Section 1]):

$$\frac{d^2y}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1-4\mu^2}{4z^2}\right)y = 0.$$

Indeed, applying the following transformation for the Meijer G-functions (see [13, Section 5.3]):

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z\right) = G_{q,p}^{n,m}\left(\begin{matrix} 1-b_1, \dots, 1-b_q \\ 1-a_1, \dots, 1-a_p \end{matrix} \middle| z^{-1}\right)$$

and taking into account (see [13, Section 6.4, (6)]) that

$$z^{1/2} G_{1,2}^{2,0}\left(\begin{matrix} a+1/2-\kappa \\ a+\mu, a-\mu \end{matrix} \middle| z\right) = z^a e^{-z/2} W_{\kappa,\mu}(z),$$

we obtain

$$I_n = n!^2 e^{-1/2} W_{-n-1/2,0}(1). \tag{28}$$

The asymptotic behavior of the Whittaker function $W_{\kappa,\mu}(z)$ for various conditions on parameters is well investigated (see, for example, [22, Chapter 4], [7, Chapter 3]). From [7, Section 7.4, (20)] we easily find that

$$W_{-n-1/2,0}(1) = \frac{e^{n-2\sqrt{n}}}{\sqrt{2} n^{n+3/4}}\left(1 + O\left(n^{-1/2}\right)\right) \quad \text{as } n \to \infty,$$

which by (28) and Lemma 2, implies the asymptotic formula for F_n.

To compute the asymptotics of q_n, we note that $q_n = n!\mathcal{L}_n(-1)$, where $\mathcal{L}_n(x) = \frac{1}{n!}(x^n e^{-x})^{(n)} = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!}$ is the Laguerre polynomial. Then the Perron asymptotic formula for the confluent hypergeometric function ${}_1F_1(a \pm n; b; z)$ [16] yields (see [24, p. 199])

$$q_n = n!\frac{e^{2\sqrt{n}}}{\sqrt[4]{n}}\left(\frac{1}{2\sqrt{\pi e}} + O\left(n^{-1/2}\right)\right),$$

and the lemma is proved. □

Theorem 1. *Let $\{q_n\}_{n\geqslant 0}$, $\{p_n\}_{n\geqslant 0}$ be defined by (21). Then $q_n \in \mathbb{Z}$, $D_n p_n \in \mathbb{Z}$ for each $n = 0, 1, 2, \ldots,$ and*

$$p_n - \gamma q_n = n!\frac{e^{-2\sqrt{n}}}{\sqrt[4]{n}}\left(\sqrt{\frac{\pi}{e}} + O\left(n^{-1/2}\right)\right), \qquad q_n = n!\frac{e^{2\sqrt{n}}}{\sqrt[4]{n}}\left(\frac{1}{2\sqrt{\pi e}} + O\left(n^{-1/2}\right)\right)$$

as $n \to \infty$.

Corollary 1. *The sequence p_n/q_n converges to Euler's constant sub-exponentially*:

$$\frac{p_n}{q_n} - \gamma = e^{-4\sqrt{n}}\left(2\pi + O\left(n^{-1/2}\right)\right) \quad \text{as } n \to \infty.$$

It is interesting to mention that the sequence of complex integrals I_n produces good rational approximations to the Euler–Gompertz constant

$$\delta := \int_0^\infty \frac{e^{-x}}{x+1}\,dx = \int_0^\infty \log(x+1)e^{-x}\,dx = 0.5963473623\ldots.$$

Theorem 2. *For each $n = 0, 1, 2, \ldots,$*

$$eI_n = q_n\delta - s_n,$$

where s_n is a solution of recurrence (22) with the initial values $s_0 = 0$, $s_1 = 1$. In particular, s_n/q_n converges to the Euler–Gompertz constant sub-exponentially:

$$\delta - \frac{s_n}{q_n} = e^{-4\sqrt{n}}\left(2\pi e + O\left(n^{-1/2}\right)\right) \quad \text{as } n \to \infty.$$

Proof. Whittaker's function $W_{\kappa,\mu}(z)$ satisfies the second-order recurrence relation (see [22, Section 2.5.1]):

$$(2\kappa - z)W_{\kappa,\mu}(z) + W_{\kappa+1,\mu}(z) - (\mu - \kappa + 1/2)(\mu + \kappa - 1/2)W_{\kappa-1,\mu}(z) = 0. \tag{29}$$

Then by (28), we easily conclude that the sequence of integrals I_n satisfies the recurrence (22). From [22, Section 5.6] and [7, Section 2.6] we have that

$$W_{-1/2,0}(1) = e^{1/2}\int_1^\infty \frac{e^{-t}}{t}\,dt = e^{-1/2}\delta, \qquad W_{1/2,0}(1) = e^{-1/2}.$$

Then from (29) we easily find that

$$W_{-3/2,0}(1) = 2e^{-1/2}\delta - e^{-1/2}.$$

For the first several values of the sequence I_n we have

$$eI_0 = e^{1/2}W_{-1/2,0}(1) = \delta = q_0\delta - s_0,$$
$$eI_1 = e^{1/2}W_{-3/2,0}(1) = 2\delta - 1 = q_1\delta - s_1,$$

with $s_0 = 0$, $s_1 = 1$ and q_0, q_1 defined in Lemma 1. Since the sequence $\{eI_n\}_{n\geqslant 0}$ satisfies the recurrence equation (22), we easily obtain that for any $n \geqslant 0$,

$$eI_n = q_n\delta - s_n,$$

which completes the proof. □

From Theorem 2 we recover a continued fraction expansion for the Euler–Gompertz constant that was first proved by Stieltjes in 1895 (see [26, Chapter 18, (92.7)]).

Corollary 2. *The Euler–Gompertz constant has the following continued fraction expansion*:

$$\delta = \frac{1}{2+} \mathop{\mathbf{K}}_{m=1}^{\infty}\left(\frac{-m^2}{2(m+1)}\right) = \frac{1}{2-}\,\frac{1^2}{4-}\,\frac{2^2}{6-}\,\frac{3^2}{8-}\cdots.$$

The nth convergent of this continued fraction has the form s_n/q_n and it slowly converges to δ to imply certain results on arithmetical nature of δ. The irrationality of the Euler–Gompertz constant δ is still an open problem. Using the representation of $\delta = eE_1(1)$ in terms of the exponential integral

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t}\,dt, \qquad |\arg z| < \pi,$$

that can be expanded in powers of z as (see [1, p. 228]),

$$E_1(z) = -\gamma - \log z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k!k}, \qquad |\arg z| < \pi,$$

we obtain the following relation connecting three famous constants e, γ, and δ,

$$-\delta = e\gamma + e\sum_{k=1}^{\infty} \frac{(-1)^k}{k!k}. \tag{30}$$

From classical results of Shidlovskii on algebraic independence of values of E-functions, it follows (see [21, Chapter 7, Theorem 1]) that the numbers e and $\sum_{k=1}^{\infty}\frac{(-1)^k}{k!k}$ are algebraically independent over $\mathbb{Q}$. Then from relation (30) we obtain that the numbers e and $\delta/e + \gamma$ are algebraically independent over $\mathbb{Q}$ and therefore we arrive at the following result due to Mahler [14].

Corollary 3. *At least one of the numbers γ, δ must be transcendental.*

5. A continued fraction for Euler's constant

Based on the results obtained in the previous two sections and using the following theorem from the general theory of continued fractions we will be able to find a continued fraction expansion (though not a simple one) for Euler's constant γ whose nth numerator and denominator coincide with the p_n and q_n, respectively. It seems to be the first non-trivial continued fraction expansion convergent to Euler's constant sub-exponentially, the elements of which can be expressed as a general pattern.

Theorem A. *(See [12, Theorem 2.2].) Let $\{A_n\}, \{B_n\}$ be sequences of complex numbers such that*

$$A_{-1}=1, \qquad A_0=b_0, \qquad B_{-1}=0, \qquad B_0=1,$$

and

$$A_nB_{n-1}-A_{n-1}B_n\neq 0, \quad n=0,1,2,\ldots.$$

Then there exists a uniquely determined continued fraction $b_0+\mathbf{K}(a_n/b_n)$ with nth numerator A_n and denominator B_n for all n. Moreover,

$$b_0=A_0, \qquad a_1=A_1-A_0B_1, \qquad b_1=B_1,$$
$$a_n=\frac{A_{n-1}B_n-A_nB_{n-1}}{A_{n-1}B_{n-2}-A_{n-2}B_{n-1}}, \qquad b_n=\frac{A_nB_{n-2}-A_{n-2}B_n}{A_{n-1}B_{n-2}-A_{n-2}B_{n-1}}, \quad n=2,3,4,\ldots.$$

Applying the above theorem to the sequences $\{p_n\}_{n\geqslant 0}$ and $\{q_n\}_{n\geqslant 0}$ we get the following.

Theorem 3. *Euler's constant γ has the following continued fraction expansion*:

$$\gamma=\mathop{\mathbf{K}}_{n=1}^{\infty}\left(a_n^*/b_n^*\right)=\frac{1}{2}\,\frac{}{-}\,\frac{1}{4}\,\frac{}{-}\,\frac{5}{16}\,\frac{}{+}\,\frac{36}{59}\,\frac{}{-}\,\frac{15\,740}{404}\,\frac{}{+}\cdots\frac{}{+}\,\frac{a_n^*}{b_n^*}\,\frac{}{+}\cdots,$$

where

$$\begin{aligned} a_1^*&=1, & a_2^*&=-1, & a_3^*&=-5, & a_4^*&=36, & a_5^*&=-15\,740,\\ b_1^*&=2, & b_2^*&=4, & b_3^*&=16, & b_4^*&=59, & b_5^*&=404, \end{aligned} \tag{31}$$

and

$$a_n^*=-\frac{(n-1)^2}{4}\Delta_n\Delta_{n-2}, \qquad b_n^*=n^2\Delta_{n-1}+\frac{(n-1)(n-2)}{2}q_{n-2}, \quad n\geqslant 6. \tag{32}$$

Here q_n is a sequence of positive integers defined by (21) and Δ_n is a sequence of integers generated by the third-order linear recurrence:

$$\begin{aligned}(n-1)(n-2)\Delta_{n+2}&=(n-2)(n+1)\left(n^2+3n-2\right)\Delta_{n+1}\\ &\quad-n^2\left(2n^3+n^2-7n-4\right)\Delta_n+(n-1)^2n^4\Delta_{n-1}, \quad n\geqslant 3,\end{aligned} \tag{33}$$

with the initial values $\Delta_1=-1$, $\Delta_2=-2$, $\Delta_3=-5$, $\Delta_4=8$. Moreover, Δ_n is positive for any $n\geqslant 4$, and Δ_{2n} is even for any $n\geqslant 1$.

Proof. Consider sequences $\{p_n\}_{n\geqslant 0}$ and $\{q_n\}_{n\geqslant 0}$ from (21) and put $p_{-1}=1$, $q_{-1}=0$. For $n\geqslant 0$, define $\eth_n := p_{n-1}q_n - p_nq_{n-1}$. Then we have

$$\eth_0 = 1, \qquad \eth_1 = \eth_2 = -1, \qquad \eth_3 = -\frac{5}{3}, \qquad \eth_4 = 2, \qquad \eth_5 = \frac{787}{5}. \tag{34}$$

From the recurrent equations (22), (23) we get the relation

$$\eth_n = (n-1)^2\eth_{n-1} + \frac{n-2}{n}q_{n-1}, \quad n\geqslant 1. \tag{35}$$

Since q_n is positive for any $n\geqslant 0$ and $\eth_4>0$, it follows easily by induction that $\eth_n$ is positive for any $n\geqslant 4$. Taking into account (34), we get $\eth_n\neq 0$ for $n=0,1,2,\dots$. Moreover, (35) implies that $n\eth_n\in\mathbb{Z}$ for any n and $n\eth_n/2\in\mathbb{Z}$ if n is even.

Now by Theorem A, we get that there exists a uniquely determined continued fraction $b_0 + \mathbf{K}(a_n/b_n)$ with nth numerator p_n and denominator q_n for all n, where $b_0=p_0=0$, $a_1=p_1-p_0q_1=1$, $b_1=q_2=1$, and

$$a_n = -\frac{\eth_n}{\eth_{n-1}}, \qquad b_n = \frac{p_{n-2}q_n - p_nq_{n-2}}{\eth_{n-1}}, \quad n\geqslant 2, \tag{36}$$

such that

$$\gamma = \mathbf{K}(a_n/b_n) = \frac{1}{2} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots. \tag{37}$$

From (35), (36) we have

$$a_n = -(n-1)^2 - \frac{n-2}{n}\cdot\frac{q_{n-1}}{\eth_{n-1}}, \quad n\geqslant 1.$$

From the recurrent equations (22), (23) we obtain

$$p_{n-2}q_n - p_nq_{n-2} = 2n\eth_{n-1} + \frac{n-2}{n}\cdot q_{n-2}$$

and hence

$$b_n = 2n + \frac{n-2}{n}\cdot\frac{q_{n-2}}{\eth_{n-1}}, \quad n\geqslant 1. \tag{38}$$

Now let $\Delta_n := n\eth_n$. Then we have $\Delta_n\in\mathbb{Z}$, Δ_n is positive for any $n\geqslant 4$ and Δ_{2n} is even for any $n\geqslant 1$. Define $\rho_0=\rho_1=\rho_2=1$, $\rho_3=3$, $\rho_4=10$,

$$\rho_n = \frac{n(n-1)\eth_{n-1}}{2} = \frac{n\Delta_{n-1}}{2}, \quad n\geqslant 5,$$

and make the equivalence transformation of the fraction (37) by the rule (see [12, Theorem 2.6])

$$a_n^* = \rho_n\rho_{n-1}a_n, \qquad b_n^* = \rho_nb_n, \quad n=1,2,3,\dots.$$

Then using first several values of the sequence q_n,

$$q_0 = 1, \qquad q_1 = 2, \qquad q_2 = 7, \qquad q_3 = 34, \qquad q_4 = 209$$

and formulas (36), (38) we get (31), (32). What is left is to show that the sequence Δ_n satisfies the recurrence (33). From (35) we have

$$q_{n-1} = \frac{\Delta_n - n(n-1)\Delta_{n-1}}{n-2}, \quad n \geqslant 3.$$

Substituting this expression in (22) we get the four-term recurrence relation (33), which completes the proof. $\square$

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References

[1] M. Abramovitz, I. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1964.
[2] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque 61 (1979) 11–13.
[3] A.I. Aptekarev (Ed.), Rational Approximation of Euler's Constant and Recurrence Relations, Current Problems in Math., vol. 9, Steklov Math. Inst. RAN, 2007 (in Russian).
[4] A.I. Aptekarev, D.N. Tulyakov, Four-term recurrence relations for γ-forms, in [3], pp. 37–43.
[5] F. Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, Bull. London Math. Soc. 11 (3) (1979) 268–272.
[6] F. Beukers, Padé approximations in number theory, in: Lecture Notes in Math., vol. 888, Springer-Verlag, Berlin, 1981, pp. 90–99.
[7] H. Buchholz, The Confluent Hypergeometric Function, Springer Tracts Nat. Philos., vol. 15, Springer-Verlag New York, Inc., 1969.
[8] S. Fischler, Irrationalité de valeurs de zêta [d'après Apéry, Rivoal, ...], Astérisque 294 (2004) 27–62, vii pp.
[9] L.A. Gutnik, On the irrationality of certain quantities involving $\zeta(3)$, Russian Math. Surveys 34 (3) (1979) 190, Acta Arith. 42 (3) (1983) 255–264.
[10] Kh. Hessami Pilehrood, T. Hessami Pilehrood, Rational approximations for the quotient of gamma values, Indag. Math. (N.S.) 20 (4) (2009) 583–601.
[11] Kh. Hessami Pilehrood, T. Hessami Pilehrood, Rational approximations for values of the digamma function and a conjecture on denominators, Math. Notes 90 (5) (2011) 730–747.
[12] W.B. Jones, W.J. Thron, Continued fractions: Analytic theory and applications, in: G.-C. Rota (Ed.), Encyclopedia Math. Appl., vol. 11, Addison–Wesley, 1980.
[13] Yu.L. Luke, The Special Functions and Their Approximations, vol. 1, Academic Press, 1969.
[14] K. Mahler, Applications of a theorem by A.B. Shidlovskii, Philos. Trans. R. Soc. Lond. Ser. A 305 (1968) 149–173.
[15] Yu.V. Nesterenko, A few remarks on $\zeta(3)$, Math. Notes 59 (6) (1996) 625–636.
[16] O. Perron, Über das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenztem Wachstum eines Parameters, J. Reine Angew. Math. 151 (1921) 63–78.
[17] M. Petkovšek, H.S. Wilf, D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA, 1997.
[18] A. van der Poorten, A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$, Math. Intelligencer 1 (4) (1978/1979) 195–203.
[19] T. Rivoal, La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs, C. R. Acad. Sci. Paris Sér. I Math. 331 (4) (2000) 267–270.
[20] T. Rivoal, Rational approximations for values of derivatives of the Gamma function, Trans. Amer. Math. Soc. 361 (2009) 6115–6149.
[21] A.B. Shidlovskii, Transcendental Numbers, de Gruyter, Berlin/New York, 1989.
[22] L.J. Slater, Confluent Hypergeometric Functions, Cambridge University Press, 1960.

[23] J. Sondow, Criteria for irrationality of Euler's constant, Proc. Amer. Math. Soc. 131 (2003) 3335–3344.
[24] G. Szegő, Orthogonal Polynomials, fourth ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[25] D.N. Tulyakov, A system of recurrence relations for rational approximations of the Euler constant, Math. Notes 85 (5) (2009) 746–750.
[26] H.S. Wall, Analytic Theory of Continued Fractions, Chelsea Pub. Company, NY, 1948.
[27] W. Zudilin, One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational, Russian Math. Surveys 56 (4) (2001) 149–150.